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Corrigenda to "Reducible Veronese surfaces"

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Abstract. We correct the definition and the list of all reducible Veronese surfaces in our previous paper "Reducible Veronese surfaces", *Adv. Geom.* **10** (2010), 719–735.

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1 Introduction

In [1] we claimed to give the complete list of reducible Veronese surfaces according to the following definition.

Definition 1. For any positive integer $n \geq 1$, we will call *reducible Veronese surface* any algebraic surface $X \subset \mathbb{P}^{n+4}(\mathbb{C})$ such that:

i) X is a non-degenerate, reduced, reducible surface of pure dimension 2;
ii) $\deg(X) = n + 3$ and $\operatorname{cod}(X) = n + 2$, so that X is a minimal degree surface;
iii) $\dim[\operatorname{Sec}(X)] \leq 4$, so that it is possible to choose a generic linear space $\mathcal{L}$ of dimension $n - 1$ in $\mathbb{P}^{n+4}$ such that $\pi_{\mathcal{L}}(X)$ is isomorphic to X, where $\pi_{\mathcal{L}}$ is the the rational projection $\pi_{\mathcal{L}} : \mathbb{P}^{n+4} \dashrightarrow \Lambda$ from $\mathcal{L}$ to a generic target $\Lambda \simeq \mathbb{P}^4$;
iv) X is connected in codimension 1, i.e. if we drop any finite number (possibly 0) of points $P_1, \ldots, P_r$ from X then $X \setminus \{P_1, \ldots, P_r\}$ is connected;
v) X is a locally Cohen–Macaulay surface.

Condition iii) deserves particular attention. When $\dim[\operatorname{Sec}(X)] \leq 4$, for a generic linear $(n-1)$-dimensional linear space $\mathcal{L}$ we have that $\pi_{\mathcal{L}|X}$ is injective. However this condition, obviously necessary, is not sufficient to get that $\pi_{\mathcal{L}|X}$ is an isomorphism. The condition $\dim[\operatorname{Sec}(X)] \leq 4$ is in fact equivalent to have that $\pi_{\mathcal{L}|X}$ is only a J-embedding

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according to the definition of Johnson (see [5], 1.2, and Proposition 1.5 of [6], chapter II, p. 37). To have that X is a reducible Veronese surface, i.e. to have that $\pi_{\mathcal{L}|X}$ is an isomorphism, instead of iii) *we need to use a different condition*:

iii)$'$ $\dim[\mathrm{Sec}(X)] \leq 4$ *and* $\dim\big[\bigcup_{x\in X}\langle T_x(X)\rangle\big] \leq 4$,

where $T_x(X)$ is the Zariski tangent space to X at x and $\langle V\rangle$ is the linear span of a variety V in a projective space. See [2] for the proof of the equivalence. From now on a reducible Veronese surface will be a surface satisfying conditions i), ii), iii)$'$, iv) and v).

Throughout [1], to get condition iii) for the members of our list, we used the condition on $\dim[\mathrm{Sec}(X)]$ and, independently, the fact that $\pi_{\mathcal{L}|X}$ has to be an isomorphism, see for instance the proof of Lemma 4. As the condition on $\dim[\mathrm{Sec}(X)]$ is necessary for iii)$'$, it follows that to classify reducible Veronese surfaces, according to the above new definition, we have to check the list of [1] and we have to exclude surfaces for which $\dim\big[\bigcup_{x\in X}\langle T_x(X)\rangle\big] \leq 4$ does not hold.

In this note we perform this check and we also fix some mistakes in the proof of Proposition 2 of [1].

2 Refining and completing the list

The list in [1] contained three types of surfaces X:

a$_n$) for any integer $n \geq 1$, a suitable union of $n+3$ planes which sits as a linearly normal scheme in $\mathbb{P}^{n+4}$ (see Definition 2 of [1] for a precise description); these surfaces were introduced in [4].

b) $X = Q \cup X_1 \cup X_2$: the union of a smooth quadric surface Q in $\mathbb{P}^3$ and two planes X_1 and X_2 sitting as a linearly normal scheme in $\mathbb{P}^5$; X_1 and X_2 cut Q, respectively, along two lines L_1, L_2, intersecting at a point $P := X_1 \cap X_2$, and $L_1 = \langle Q\rangle \cap X_1$, $L_2 = \langle Q \cup X_1\rangle \cap X_2$.

c) $X = Q \cup X_1 \cup X_2$: the union of a smooth quadric surface Q in $\mathbb{P}^3$ and two planes X_1 and X_2, sitting as a linearly normal scheme in $\mathbb{P}^5$; X_1, X_2 and Q intersect pairwise transversally along a unique line $L := Q \cap X_1 \cap X_2$ and $L = \langle Q\rangle \cap X_1 \cap X_2$.

It is easy to see that $\dim\big[\bigcup_{x\in X}\langle T_x(X)\rangle\big] \leq 4$ in both cases a$_n$) and b). In contrast, if we consider points $x \in L$ in case c), the tangent space at x to X is $\langle T_x(Q) \cup X_1 \cup X_2\rangle \simeq \mathbb{P}^4$ and $\bigcup_{x\in L}\langle T_x(Q) \cup X_1 \cup X_2\rangle = \mathbb{P}^5$, so that there is no point $\mathcal{L} \in \mathbb{P}^5$ such that $\pi_{\mathcal{L}|X}$ is an isomorphism.

Unfortunately, there exist two other surfaces to check, i.e. two surfaces satisfying conditions i), ii), iii), iv), v) but not considered in [1]. These surfaces sit as linearly normal schemes, respectively, in $\mathbb{P}^5$ and $\mathbb{P}^6$:

d) $X = S \cup X_1$ where S is a smooth rational cubic scroll in $\mathbb{P}^4$ having a line L as fundamental section and X_1 is a plane such that $S \cap X_1 = \langle S\rangle \cap X_1 = L$.

e) $X = S \cup X_1 \cup X_2$ where $S \cup X_1$ is a surface as in d) and X_2 is a plane such that $S \cap X_1 \cap X_2 = \langle S \cup X_1\rangle \cap X_2 = L$.

Obviously conditions i), ii) and iv) are satisfied. Condition v) is satisfied by arguing as in Lemma 1 of [1]. For a surface X as in d) we have $\dim[\mathrm{Sec}(X)] \leq 4$ by direct calculation with a computer algebra system or by considering that every line joining generic points of S and X_1 is contained in the 4-dimensional quadric cone having X_1 as vertex and the smooth conic Γ as base, where Γ is the smooth conic generating S with L. For a surface X as in e) we have $\dim[\mathrm{Sec}(X)] \leq 4$ by looking at every pair of irreducible components of X.

A surface X as in d) can also be isomorphically projected in $\mathbb{P}^4$ because one has $\dim\left[\bigcup_{x\in X}\langle T_x(X)\rangle\right] \leq 4$. In contrast, if we consider points $x \in L$ in case e), the tangent space at x to X is $\langle T_x(S) \cup X_1 \cup X_2\rangle \simeq \mathbb{P}^4$ and $\bigcup_{x\in L}\langle T_x(S) \cup X_1 \cup X_2\rangle$ is a quadric cone in $\mathbb{P}^6$, so that its dimension is 5, hence, for any line $\mathcal{L} \in \mathbb{P}^6$, $\pi_{\mathcal{L}|X}$ cannot be an isomorphism.

Now we prove that there are no other reducible Veronese surfaces up to those above. In Proposition 2 of [1] we claimed that every irreducible component of a reducible Veronese surface X can be only a plane, a smooth quadric in $\mathbb{P}^3$ or a quadric in $\mathbb{P}^3$ having rank 3. With this assumption we get only the surfaces a$_n$), b), c) as it is proved in [1]. However there are other possibilities for the irreducible components of X: by Theorem 1 of [1], they are reduced surfaces of minimal degree in their spans, and the classification of such surfaces is quoted in Theorem 0.1 of [3] where ”rational normal scroll” for 2-dimensional varieties means: a smooth rational normal scroll or a cone over a smooth rational normal curve. Not all these surfaces were well considered in Proposition 2 of [1], so we have to fill this gap.

Let us consider cones Y over smooth rational normal curves and let E be the vertex of a cone Y. The tangent space at E to Y, which is $\langle Y\rangle$, cannot have dimension bigger than 4 otherwise condition iii)$'$ would be not satisfied, so that $\deg(Y) \leq 3$. If $\deg(Y) = 2$ the other irreducible components of X must be planes (see the final part of the proof of Proposition 2 in [1]) and the union of a rank 3 quadric cone in $\mathbb{P}^3$ and planes can be excluded by arguing as in Case 1) of the proof of Theorem 3 in [1]. It follows that here we have to consider only the case $\deg(Y) = 3$. By contradiction, let us assume that an irreducible component of a reducible Veronese surface X is a degree 3 cone Y as above, having vertex E. Let X_i be another component of X. To satisfy condition iii)$'$ we must have $E \notin X_i$ so that $Y \cap X_i = \langle Y\rangle \cap \langle X_i\rangle$ is a single point $P \in Y$, $P \neq E$, by Corollary 2 of [1]. If X_i is not a plane, the join of Y and X_i has dimension 5, hence $\dim[\mathrm{Sec}(X)] \geq 5$, which is a contradiction. If X_i is a plane, any projection $\pi_{\mathcal{L}}$ of $Y \cup X_i$ in $\mathbb{P}^4$ cannot be an isomorphism because $\pi_{\mathcal{L}}(Y) \cap \pi_{\mathcal{L}}(X_i)$ cannot be a single point.

Now let us consider smooth rational normal scrolls of dimension 2. As no smooth surface can be isomorphically projected in $\mathbb{P}^4$ with the exception of the Veronese surface, we have to consider only smooth rational cubic scrolls S in $\mathbb{P}^4$ (other than smooth quadrics in $\mathbb{P}^3$ examined in [1]). In spite of what we said in the proof of Proposition 2 of [1], p. 126, lines 13–18, also a smooth rational cubic scroll S in $\mathbb{P}^4$ can be an irreducible component of a reducible Veronese surface X. The correct part of the proof of Proposition 2 in [1] shows that this is possible only when all other components of X are planes cutting $\langle S\rangle$ and S only along a line L which is its fundamental section. This line escaped the analysis made in [1], where only the fibres of the scroll were considered. All other possibilities, involving planes and quadrics, are considered and correctly excluded in Proposition 2 of [1].

As we have seen, the union of a smooth cubic scroll S in $\mathbb{P}^4$ and one or two planes, cutting $\langle S\rangle$ and S along its fundamental section L, gives rise to two surfaces to be checked. No other plane can be admitted by Lemma 3 of [1] and condition iii)$'$.

In conclusion: the surfaces a$_n$), b) and d) can be isomorphically projected in $\mathbb{P}^4$, but not c) and e). This is the complete list of reducible Veronese surfaces with the correct condition iii)$'$ instead of iii).

Remark 1. This note is also a correction of the list of reducible Veronese surfaces quoted in Theorem 1 of [2] and never used in that paper.

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